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SELECTIVE GENERALIZED COORDINATES PARTITIONING METHOD FOR MULTIBODY SYSTEMS WITH NON-HOLONOMIC CONSTRAINTS

Ayman A. Nada*
Benha Faculty of Engineering,
Benha University, Egypt.
13512-Benha, Qalubia, Egypt.
ayman.nada@bhit.bu.edu.eg

Abdullateef H. Bashiri
College of Engineering,
Jazan University, KSA.
PO.706 - 442502 Jazan, KSA.
absheiri@jazanu.edu.sa

ABSTRACT
The goal of this research work is to extend the method of generalized coordinates partitioning to include both holonomic and nonholonomic constraints. Furthermore, the paper proposes a method for selective coordinates for integration instead of identifying a set of independent coordinates at each integration step. The effectiveness of the proposed method is presented and compared with full-coordinates integration as well as generalized coordinates partitioning method. The proposed method can treat large-scale systems as one of the main advantages of multi-body systems.

Keywords:Nonholonomic systems, coordinates partitioning.

1 INTRODUCTION
In some mechanical systems, one encounters various types of constraints that restrict relative motions between interconnected bodies of the system. These constraints may be given by algebraic equations that are connecting coordinates (holonomic or geometric constraints), or by differential equations which restrict some components of velocities (kinematic constraints). Nonintegrable kinematic constraints, which cannot be reduced to holonomic ones, are called nonholonomic constraints.

In the case of systems subjected to holonomic constraints, which present most of the geometric joints (rigid, revolute, prismatic, etc.,); the constraints equations are integrable. Many effective methods of formulating and solving differential equations of motion of multibody systems with holonomic constraints have been presented in the literature [1–6].

On the other hand, non-holonomic constraints impose no restriction on the position level of the system and must be satisfied at the velocity and acceleration levels. In the numerical solution algorithm, holonomic and non-holonomic constraints equations must be differently treated [7]. Examples of non-holonomic constraints are the condition of systems with meshing gears and pure rolling. Figure (1), shows a model of the Epicyclic gear train, the holonomic constraints include the rigid joint between the ring gear and ground, revolute joints between the sun gear and ground, and between planetary gears and carrier arm. The non-holonomic constraints include the zero relative velocity between meshing gears at contact points.

In the case of large-scale multibody systems, and to avoid numerical and configuration singularities, generalized coordinates partitioning for holonomic constraints has been provided numerically [6]. Based on Gaussian elimination algorithm, a set of independent generalized coordinates can be identified numerically at each step and integrated with time. The other set of dependent coordinates can be obtained by iterative Newton-Raphson algorithm, which is used to stabilize the coordinates and ensure that the holonomic constraint equations are satisfied at the position level. This position analysis step does not include the non-holonomic constraint equations, which make the number of dependent coordinates less than the number of dependent velocities.

*On leave to, College of Engineering, Jazan University, Jazan, KSA.

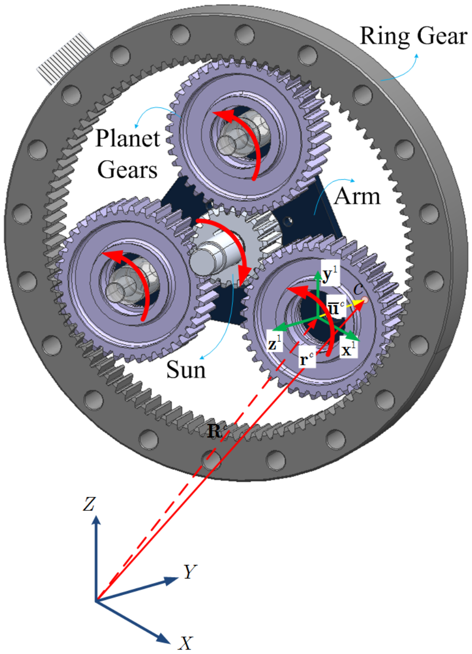

FIGURE 1. Planetary Gearbox as a Multibody system

In this paper, the generalized coordinates partitioning algorithm is extended to define dependent generalized velocity coordinates associated with nonholonomic constraints. Furthermore, a method for selective generalized coordinates partitioning is proposed for numerical integration of multibody systems subjected to non-holonomic constraints. This method can add significant features to the generalized coordinates partitioning regarding the calculation time of the simulation process. A comparison between the numerical integration algorithms of nonholonomic systems is carried out. Although the equations of motion have been presented in the general form, but the simulation work concerns a planetary gear train as an example of such systems.

2 HOLONOMIC MULTIBODY SYSTEMS

The mixed system of differential equations and kinematic relationships are used to define the acceleration vector and the vector of Lagrange multipliers of the multi-rigid body systems. In the case of a multibody system with holonomic constraints equations, which are nonlinear algebraic constraint equations that represent the joints and the specified motion trajectories. The constraints equations and its derivatives can be written as follows [3]:

$$\mathbf{C}(\mathbf{q},t) = \mathbf{0} \tag{1}$$
$$\dot{\mathbf{C}}(\mathbf{q},t) = \mathbf{C}_q\dot{\mathbf{q}} + \mathbf{C}_t = \mathbf{0} \tag{2}$$
$$\ddot{\mathbf{C}}(\mathbf{q},t) = \mathbf{C}_q\ddot{\mathbf{q}} + (\mathbf{C}_q\dot{\mathbf{q}})_q\dot{\mathbf{q}} + 2\mathbf{C}_{qt}\dot{\mathbf{q}} + \mathbf{C}_{tt} = 0 \tag{3}$$

where $\mathbf{q}$ is the generalized coordinates and expressed as $\mathbf{q} = \left[\mathbf{R}^T\ \theta^T\right]^T$, such that $\mathbf{R} = \left[R_x\ R_y\ R_z\right]^T$ is the Cartesian coordinates of the body origin and $\theta = \left[\theta_1\ \theta_2\ \theta_3\ \theta_4\right]^T$ is the Euler parameters that present the rotational coordinates of the body [3]. Euler parameters are used in this paper in order to avoid the kinematic singularity associated with the three-parameter representation of the rotation [8]. The matrix $\mathbf{C}_\mathbf{q} = \frac{\partial \mathbf{C}(\mathbf{q},t)}{\partial \mathbf{q}}$ is the Jacobian matrix of the kinematic constraints function $\mathbf{C}$. The vectors $\mathbf{C}_t$, $\mathbf{C}_{\mathbf{q}t}$, $\mathbf{C}_{tt}$ in Eqs.(2,3) are zero vectors if the kinematic constraints are *scleronomic* which indicates that the kinematic constraints are not dependent on time t explicitly. Description of spatial joint constraints in holonomic multibody systems has been presented and discussed in many research work. Not only this, but also the Jacobian matrix associated with each joint constraints [9], therefore, the terms of Eqs.(1 - 3) can be easily constructed.

The equations of motion of a system of interconnected bodies in its invertible form, i.e., involve the Euler parameters dependency, can be expressed as [11]:

$$\begin{bmatrix}\hat{\mathbf{M}}^i & \hat{\mathbf{C}}_\mathbf{q}^{iT}\\ \hat{\mathbf{C}}_\mathbf{q}^i & \mathbf{0}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{p}}^i\\ \lambda^i\end{bmatrix} = \begin{bmatrix}\hat{\mathbf{Q}}^i\\ \mathbf{Q}_d^i\end{bmatrix} \tag{4}$$

such that $\ddot{\mathbf{p}}^i = \left[\ddot{\mathbf{q}}^{iT}\ \lambda^{i\theta}\right]^T$ where $\lambda^{i\theta}$ is the Lagrange multiplier associated with Euler parameters constraint. The constraints Jacobian $\hat{\mathbf{C}}_q = \left[\mathbf{C}_R\ \mathbf{C}_\theta\ \mathbf{0}\right]$ is obtained by modifying the constraints Jacobian $\mathbf{C}_q = \left[\mathbf{C}_R\ \mathbf{C}_\theta\right]$ by inserting a number of zero columns associated with λ^θ. The associated generalized force vector is expressed as:

$$\hat{\mathbf{Q}}^i = \left[\mathbf{Q}^{iT}\ \mathbf{Q}_d^{\theta iT}\right]^T \tag{5}$$

The vector $\mathbf{Q}^i = \mathbf{Q}_e^i + \mathbf{Q}_v^i$ include the external forces as well as the quadratic velocity vector, which includes the Coriolis and centrifugal forces. The vector $\mathbf{Q}_d$ is defined as:

$$\mathbf{Q}_d = \mathbf{C}_\mathbf{q}\ddot{\mathbf{q}} = -\left(\mathbf{C}_\mathbf{q}\dot{\mathbf{q}}\right)_\mathbf{q}\dot{\mathbf{q}} - 2\mathbf{C}_{\mathbf{q}t}\dot{\mathbf{q}} - \mathbf{C}_{tt} \tag{6}$$

In this form, the matrix $\hat{\mathbf{M}}^i$ is the extended mass matrix of body i, which can be written as:

$$\underset{[8\times 8]}{\hat{\mathbf{M}}^i} = \begin{bmatrix}\mathbf{M}_{\mathbf{RR}}^i & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}_{\theta\theta}^i & C_\theta^{\theta iT}\\ \mathbf{0} & C_\theta^{\theta i} & \mathbf{0}\end{bmatrix} \tag{7}$$

If the vector of generalized external forces $\mathbf{Q}_e^i$ is known, the unknowns in Eq.(4) are the vector of accelerations $\ddot{\mathbf{p}}^i$ and the vector of Lagrange multipliers λ^i.

It should be mentioned here that only the differentiated form of the constraint Eq.(6) is satisfied, and does not guarantee that $\mathbf{C}(\mathbf{q},t) = \mathbf{0}$, and $\dot{\mathbf{C}}(\mathbf{q},t) = \mathbf{0}$ to be true as time integration goes on; this means that some constraint drifting at the position and velocity level may take place. Therefore after each integration step relying on accelerations, it is necessary to correct the systems state by moving positions and velocities back to their manifolds, $\mathbf{C}(\mathbf{q},t)$, $\dot{\mathbf{C}}(\mathbf{q},t)$ respectively, using small corrections. This corrections process is called post stabilization procedure and can be implemented using Newton-Raphson method.

2.1 Position Stabilization

After each integration step, it is necessary to correct the systems state by moving positions back to their manifolds, $\mathbf{C}(\mathbf{q},t) = \mathbf{0}$, by means of small corrections. The stabilization step takes the result of the integration step as input and gives a correction so that the final result is closer to the constraint manifold. The position stabilization is then, can be used using Newton-Raphson method, as follows:

1. At each integration step time t, the constraints function, $\mathbf{C}(\mathbf{q},t)$ as well as the constraints Jacobian, $\mathbf{C}_{\mathbf{q}} = \frac{\partial \mathbf{C}(\mathbf{q},t)}{\partial \mathbf{q}}$ should be evaluated,
2. Evaluate the Newton difference, $\Delta \mathbf{q}_n$ as:

$$\Delta \mathbf{q}_n = -\mathbf{C}_{\mathbf{q}}^{-1}(\mathbf{q}_n)\,\mathbf{C}(\mathbf{q}_n) \tag{8}$$

3. Update the generalized coordinate as follows:

$$\mathbf{q}_{n+1} = \mathbf{q}_n + \Delta \mathbf{q}_n \tag{9}$$

4. When the error limit is obeyed, i.e. $\|\Delta \mathbf{q}_n\| \le \varepsilon$, then $\mathbf{q}(t) = \mathbf{q}_{n+1}$.

Many algorithms have been developed in the literature to estimate the inverse of constraints Jacobian [12]. The Jacobian matrix is full row rank and the right pseudoinverse matrix which is defined as $C_q^{-1} = C_q^T \left(C_q C_q^T\right)^{-1}$ can be applied. Note that, in MATLAB, the command *pinv* returns the pseudoinverse of the non-square matrix using the least square method [12].

2.2 Velocity Stabilization

1. At each integration step time t, the constraints function derivative, $\dot{\mathbf{C}}(\mathbf{q},t)$ as well as the constraints Jacobian, $\mathbf{C}_{\mathbf{q}}$ should be evaluated. Note that the vector $\dot{\mathbf{C}}(\mathbf{q},t)$ can be obtained using Eq.(2) and $\frac{\partial \dot{\mathbf{C}}(\mathbf{q},t)}{\partial \dot{\mathbf{q}}} = \frac{\partial}{\partial \dot{\mathbf{q}}}\left(\mathbf{C}_q \dot{\mathbf{q}} + \mathbf{C}_t\right) = \mathbf{C}_{\mathbf{q}}$.
2. Evaluate the Newton difference, $\Delta \dot{\mathbf{q}}_n$ as:

$$\Delta \dot{\mathbf{q}}_n = -\mathbf{C}_{\mathbf{q}}^{-1}(\mathbf{q}_n)\,\dot{\mathbf{C}}(\mathbf{q}_n,t) \tag{10}$$

3. Update the generalized velocities as follows:

$$\dot{\mathbf{q}}_{n+1} = \dot{\mathbf{q}}_n + \Delta \dot{\mathbf{q}}_n \tag{11}$$

4. Since the constraints function derivative is linear in generalized velocities; no iteration is required, and the pervious procedure is called one step iteration.

2.3 Generalized Coordinate Partitioning

This section presents the basic method of generalized coordinate partitioning that has been addressed in many previous publications [2,3,6]. In this method, a Gaussian elimination algorithm with full pivoting decomposes the constraint Jacobian matrix, identifies dependent variables, and constructs an influence coefficient matrix relating variations independent and independent variables [2].

For the systems subjected to holonomic constraints, if the constraints equations $\mathbf{C}(\mathbf{q},t) = \mathbf{0}$ are linearly independent, the Jacobian matrix of the kinematic constraints $[\mathbf{C}_{\mathbf{q}}]_{n_c \times n_q} = \frac{\partial \mathbf{C}(\mathbf{q},t)}{\partial \mathbf{q}}$ has a full row rank, where n_q is the number of generalized coordinates and n_c is the number of constraint equations. In this case, at a value of $\mathbf{q}$ that satisfies $\mathbf{C}(\mathbf{q},t) = \mathbf{0}$**,** Now, the system generalized coordinates can be partitioned into

$$\mathbf{q} = \left[\,\mathbf{q}_d^T\ \mathbf{q}_i^T\,\right]^T \tag{12}$$

Where $\mathbf{q}_d$ is the set of dependent coordinates, and $\mathbf{q}_i$ is the set of independent coordinates. Consider a virtual displacement $\delta \mathbf{q}$ that satisfies the constraint equations, $\mathbf{C}(\mathbf{q},t) = \mathbf{0}$, to first order; that is $\mathbf{C}_{\mathbf{q}} \delta \mathbf{q} = \mathbf{0}$**.** The Gaussian reduction technique may be employed to transform this equation to the form of

$$\frac{\partial \mathbf{C}}{\partial \mathbf{q}_d}\delta \mathbf{q}_d + \frac{\partial \mathbf{C}}{\partial \mathbf{q}_i}\delta \mathbf{q}_i = 0 \tag{13}$$

$$\Updownarrow$$

$$\mathbf{C}_{\mathbf{q_d}}\delta \mathbf{q}_d + \mathbf{C}_{\mathbf{q}_i}\delta \mathbf{q}_i = 0 \tag{14}$$

such that $\mathbf{q}_i$ and $\mathbf{q}_d$ are two vectors having $(n_q - n_c)$ and n_c components, respectively. The dimensions of the Jacobian submatrices can be expressed as $[\mathbf{C}_{\mathbf{q}_i}]_{n_c \times (n_q - n_c)}$ and $[\mathbf{C}_{\mathbf{q}d}]_{n_c \times n_c}$. Thus, Eq. (14) can be solved for $\delta \mathbf{q}_d$ as:

$$\delta \mathbf{q}_d = -\mathbf{C}_{\mathbf{q}_d}^{-1} \mathbf{C}_{\mathbf{q_i}} \delta \mathbf{q}_i \tag{15}$$

It is concluded that the virtual change in the set of dependent coordinates $\delta \mathbf{q}_d$ can be expressed in terms of the change in accompanied set of independent coordinates $\delta \mathbf{q}_i$. The virtual changes in the total vector of system coordinates can be written in terms of the virtual changes of the independent coordinates as

$$\delta \mathbf{q} = \begin{bmatrix} \delta \mathbf{q}_d \\ \delta \mathbf{q}_i \end{bmatrix} = \begin{bmatrix} -\mathbf{C}_{\mathbf{q}_d}^{-1} \mathbf{C}_{\mathbf{q_i}} \\ \mathbf{I} \end{bmatrix} \delta \mathbf{q}_i = \mathbf{B}\ \delta \mathbf{q}_i \tag{16}$$

Where $\mathbf{B}$ is the velocity transformation matrix and can be expressed with dimensions $\mathbf{B}_{n_q \times (n_q - n_c)}$. Note that $n_F = (n_q - n_c)$ is the number of degrees of freedom of the system, by which the matrix dimensions are $[-\mathbf{C}_{\mathbf{q}_d}^{-1} \mathbf{C}_{\mathbf{q_i}}]_{n_c \times n_F}$, identity matrix $\mathbf{I}_{n_F \times n_F}$, consequently, the matrix $\mathbf{B}_{n_q \times nF}$, that can be defined as

$$\mathbf{B} = \begin{bmatrix} -\mathbf{C}_{\mathbf{q}_d}^{-1} \mathbf{C}_{\mathbf{q_i}} \\ \mathbf{I} \end{bmatrix} \tag{17}$$

In the case of holonomic systems, in which the joint constraints are not explicit functions of time. Thus, one can conclude that

$$\dot{\mathbf{q}} = \mathbf{B}\ \dot{\mathbf{q}}_i \tag{18}$$

The matrix $\mathbf{B}$ defines the relationship between the generalized velocities, i.e., the total vector of the system velocities and a smaller independent subset of this vector.

Elimination of Lagrange multipliers: An important property is that the constraint forces for independent coordinates should be equal zero. This property can be proved as follows

$$\begin{aligned} \mathbf{B}^T \mathbf{C}_q^T &= \begin{bmatrix} -\mathbf{C}_{\mathbf{q_i}}^T \mathbf{C}_{\mathbf{q}_d}^{-T} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{C}_{\mathbf{q}_d}^T \\ \mathbf{C}_{\mathbf{q}_i}^T \end{bmatrix} \\ &= -\mathbf{C}_{\mathbf{q_i}}^T \mathbf{C}_{\mathbf{q}_d}^{-T} \mathbf{C}_{\mathbf{q}_d}^T + \mathbf{C}_{\mathbf{q}_i}^T \\ &= -\mathbf{C}_{\mathbf{q_i}}^T + \mathbf{C}_{\mathbf{q}_i}^T = \mathbf{0} \end{aligned}$$

and thus, one can conclude that

$$\mathbf{B}^T \mathbf{C}_q^T \lambda = \mathbf{0} \tag{19}$$

Which implies that the vector of generalized constraint forces $\mathbf{C}_q^T \lambda$ is orthogonal to the columns of the velocity transformation matrix $\mathbf{B}$. Thus, if the equation of motion $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}_{\mathbf{q}}^T \lambda = \mathbf{Q}$ is premultiplied by $\mathbf{B}^T$, one obtains

$$\mathbf{B}^T \mathbf{M} \ddot{\mathbf{q}} + \mathbf{B^T} \mathbf{C}_{\mathbf{q}}^T \lambda = \mathbf{B^T Q}$$

Using the orthogonality condition of Eq.(19), yields

$$\mathbf{B}^T \mathbf{M} \ddot{\mathbf{q}} = \mathbf{B^T Q} \tag{20}$$

Equation (20) presents the reduced model of the equation of motion by eliminating the constraint forces and the associated Lagrange multipliers.

Recall that the constraint equations is defined in the acceleration level by Eq.(3), which yields that $\mathbf{C}_q \ddot{\mathbf{q}} = \mathbf{Q}_d$, therefore one can write Eq.(6) as

$$\mathbf{C}_{\mathbf{q_d}} \ddot{\mathbf{q}}_d + \mathbf{C}_{\mathbf{q}_i} \ddot{\mathbf{q}}_i = \mathbf{Q}_d \tag{21}$$

Thus, the dependent acceleration can be calculated as

$$\ddot{\mathbf{q}}_d = -\mathbf{C}_{\mathbf{q_d}}^{-1} \mathbf{C}_{\mathbf{q}_i} \ddot{\mathbf{q}}_i + \mathbf{C}_{\mathbf{q_d}}^{-1} \mathbf{Q}_d \tag{22}$$

Therefore, the generalized acceleration vector can be written as

$$\ddot{\mathbf{q}} = \begin{bmatrix} \ddot{\mathbf{q}}_d \\ \ddot{\mathbf{q}}_i \end{bmatrix} = \begin{bmatrix} -\mathbf{C}_{\mathbf{q_d}}^{-1} \mathbf{C}_{\mathbf{q}_i} \ddot{\mathbf{q}}_i + \mathbf{C}_{\mathbf{q_d}}^{-1} \mathbf{Q}_d \\ \ddot{\mathbf{q}}_i \end{bmatrix} = \begin{bmatrix} -\mathbf{C}_{\mathbf{q_d}}^{-1} \mathbf{C}_{\mathbf{q}_i} \\ \mathbf{I} \end{bmatrix} \ddot{\mathbf{q}}_i + \begin{bmatrix} \mathbf{C}_{\mathbf{q_d}}^{-1} \mathbf{Q}_d \\ \mathbf{0} \end{bmatrix}$$

In compact form, $\ddot{\mathbf{q}}$ can be expressed as

$$\ddot{\mathbf{q}} = \mathbf{B}\ \ddot{\mathbf{q}}_i + \mho \tag{23}$$

in which the generalized accelerations is expressed in terms of the independent accelerations only plus the $\mho$ term. the vector $\mho$ is defined as

$$\underset{n_q\times 1}{\mho} = \begin{bmatrix} \mathbf{C}_{\mathbf{q_d}}^{-1}\mathbf{Q}_d \\ \mathbf{0} \end{bmatrix}$$

Substituting Eq.(23) into Eq.(20), yields

$$\mathbf{B}^T\mathbf{M}\mathbf{B}\,\ddot{\mathbf{q}}_i = \mathbf{B}^{\mathbf{T}}\mathbf{Q} - \mathbf{B}^T\mathbf{M}\mho \tag{24}$$

Thus, the equations of motion in terms of the independent accelerations can be expressed as

$$\bar{\mathbf{M}}\,\ddot{\mathbf{q}}_i = \bar{\mathbf{Q}} \tag{25}$$

where $\bar{\mathbf{M}}$ and $\bar{\mathbf{Q}}$ are defined as

$$\underset{n_F\times n_F}{\bar{\mathbf{M}}} = \mathbf{B}^T\mathbf{M}\mathbf{B} \tag{26}$$

$$\underset{n_F\times 1}{\bar{\mathbf{Q}}} = \mathbf{B}^{\mathbf{T}}\mathbf{Q} - \mathbf{B}^T\mathbf{M}\mho \tag{27}$$

The inverse of the matrix $\bar{\mathbf{M}}$ does exist for a well-posed problem, and Eq.(25) can be used to calculate the independent accelerations that can be integrated forward in time in order to determine the independent coordinates and velocities. The dependent coordinates, velocities, and accelerations can be obtained by using Eqs. $\mathbf{C}(\mathbf{q},t) = \mathbf{0}$, $\dot{\mathbf{C}}(\mathbf{q},t) = \mathbf{0}$, and Eq.(22), respectively.

Now, it is obvious that the system singularities due to Euler parameters dependency can be avoided by implementing the coordinates partitioning algorithm, i.e., the extended form of the equations of motion, Eq.(7) is worthless, and system equations can be expressed in terms of the nominal mass matrix. The explicit form of the reduced order mass matrix can be presented as

$$\begin{aligned}
\underset{n_F\times n_F}{\bar{\mathbf{M}}} &= \begin{bmatrix} -\mathbf{C}_{\mathbf{q_i}}^T\mathbf{C}_{\mathbf{q}_d}^{-T} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{M}}_{dd} & \bar{\mathbf{M}}_{id} \\ \bar{\mathbf{M}}_{di} & \bar{\mathbf{M}}_{ii} \end{bmatrix} \begin{bmatrix} -\mathbf{C}_{\mathbf{q}_d}^{-1}\mathbf{C}_{\mathbf{q_i}} \\ \mathbf{I} \end{bmatrix} \\
&= \begin{bmatrix} -\mathbf{C}_{\mathbf{q_i}}^T\mathbf{C}_{\mathbf{q}_d}^{-T}\bar{\mathbf{M}}_{dd} + \bar{\mathbf{M}}_{di} & -\mathbf{C}_{\mathbf{q_i}}^T\mathbf{C}_{\mathbf{q}_d}^{-T}\bar{\mathbf{M}}_{id} + \bar{\mathbf{M}}_{ii} \end{bmatrix} \begin{bmatrix} -\mathbf{C}_{\mathbf{q}_d}^{-1}\mathbf{C}_{\mathbf{q_i}} \\ \mathbf{I} \end{bmatrix} \\
&= \mathbf{C}_{\mathbf{q_i}}^T\mathbf{C}_{\mathbf{q}_d}^{-T}\bar{\mathbf{M}}_{dd}\mathbf{C}_{\mathbf{q}_d}^{-1}\mathbf{C}_{\mathbf{q_i}} - \bar{\mathbf{M}}_{di}\mathbf{C}_{\mathbf{q}_d}^{-1}\mathbf{C}_{\mathbf{q_i}} - \mathbf{C}_{\mathbf{q_i}}^T\mathbf{C}_{\mathbf{q}_d}^{-T}\bar{\mathbf{M}}_{id} + \bar{\mathbf{M}}_{ii}
\end{aligned}$$

Also, the explicit form of the force vector can be presented as

$$\begin{aligned}
\underset{n_F\times 1}{\bar{\mathbf{Q}}} &= \begin{bmatrix} -\mathbf{C}_{\mathbf{q_i}}^T\mathbf{C}_{\mathbf{q}_d}^{-T} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{Q_{dd}} \\ \mathbf{Q_{ii}} \end{bmatrix} - \begin{bmatrix} -\mathbf{C}_{\mathbf{q_i}}^T\mathbf{C}_{\mathbf{q}_d}^{-T} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{M}}_{dd} & \bar{\mathbf{M}}_{id} \\ \bar{\mathbf{M}}_{di} & \bar{\mathbf{M}}_{ii} \end{bmatrix} \begin{bmatrix} \mathbf{C}_{\mathbf{q_d}}^{-1}\mathbf{Q}_d \\ \mathbf{0} \end{bmatrix} \\
&= -\mathbf{C}_{\mathbf{q_i}}^T\mathbf{C}_{\mathbf{q}_d}^{-T}\mathbf{Q_{dd}} + \mathbf{Q_{ii}} \\
&\quad - \begin{bmatrix} -\mathbf{C}_{\mathbf{q_i}}^T\mathbf{C}_{\mathbf{q}_d}^{-T}\bar{\mathbf{M}}_{dd} + \bar{\mathbf{M}}_{di} & -\mathbf{C}_{\mathbf{q_i}}^T\mathbf{C}_{\mathbf{q}_d}^{-T}\bar{\mathbf{M}}_{id} + \bar{\mathbf{M}}_{ii} \end{bmatrix} \begin{bmatrix} \mathbf{C}_{\mathbf{q_d}}^{-1}\mathbf{Q}_d \\ \mathbf{0} \end{bmatrix} \\
&= -\mathbf{C}_{\mathbf{q_i}}^T\mathbf{C}_{\mathbf{q}_d}^{-T}\mathbf{Q_{dd}} + \mathbf{Q_{ii}} + \mathbf{C}_{\mathbf{q_i}}^T\mathbf{C}_{\mathbf{q}_d}^{-T}\bar{\mathbf{M}}_{dd}\mathbf{C}_{\mathbf{q_d}}^{-1}\mathbf{Q}_d - \bar{\mathbf{M}}_{di}\mathbf{C}_{\mathbf{q_d}}^{-1}\mathbf{Q}_d
\end{aligned}$$

3 NON-HOLONOMIC MULTIBODY SYSTEMS

Holonomic and, in addition, non-holonomic constraints may exist in mechanical systems. Due to the existence of holonomic and nonholonomic constraints, the system generalized coordinates, velocities, and accelerations must satisfy certain kinematical relationships in addition to the dynamical differential equations of motion. These kinematic relationships, which may be linear or nonlinear functions in the system generalized coordinates and velocities, must be satisfied throughout the dynamic motion of the multibody system.

The constraint equations of a system that subjected holonomic and nonholonomic constraints can be written as:

$$\mathbf{C}^h(\mathbf{q},t) = \mathbf{0} \tag{28}$$

$$\mathbf{C}^{nh}(\mathbf{q},\dot{\mathbf{q}},t) = \mathbf{0} \tag{29}$$

If the nonholonomic constraints are linear in the generalized velocities, then Eq.(29) can be written as:

$$\mathbf{C}^{nh}(\mathbf{q},\dot{\mathbf{q}},t) = \mathbf{G}(\mathbf{q},t)\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q},t) = \mathbf{0} \tag{30}$$

$\mathbf{C}^h$ and $\mathbf{C}^{nh}$ are respectively the vector functions of holonomic and nonholonomic constraint equations, $\mathbf{G}$ and $\mathbf{g}$ are respectively a matrix and a vector that may depend on the system coordinates and time. It is important to emphasize that the nonholonomic constraint equations in Eq.(30) must be nonintegrable and may not be reducible to an integrable form by virtue of the other constraints [1].

These nonholonomic constraint equations impose restrictions on the generalized velocities and consequently restrict velocities and accelerations. They do not, however, impose any restrictions on the generalized coordinates. This implies that the number of independent coordinates is larger than the number of the independent velocities or accelerations.

The system Jacobian matrix corresponding to both holonomic and nonholonomic constraints can be written as:

$$\mathbf{C}_{(\mathbf{q},\dot{\mathbf{q}})} = \begin{bmatrix} \frac{\partial \mathbf{C}^h(\mathbf{q},t)}{\partial \mathbf{q}} \\ \frac{\partial \mathbf{C}^{nh}(\mathbf{q},\dot{\mathbf{q}},t)}{\partial \dot{\mathbf{q}}} \end{bmatrix} = \begin{bmatrix} \mathbf{C}^h_{\mathbf{q}} \\ \mathbf{C}^{nh}_{\dot{\mathbf{q}}} \end{bmatrix} = \begin{bmatrix} \mathbf{C}^h_{\mathbf{q}} \\ \mathbf{G}(\mathbf{q},t) \end{bmatrix} \tag{31}$$

Where $\mathbf{C}^h_{\mathbf{q}}$ is the Jacobian matrix of the holonomic constraints, $\mathbf{C}^{nh}_{\dot{\mathbf{q}}} = \mathbf{G}(\mathbf{q},t)$ is the Jacobian matrix of the nonholonomic constraints. The time derivative of the holonomic constraints can be re-written by modifying Eq.(2) as:

$$\dot{\mathbf{C}}^h(\mathbf{q},\dot{\mathbf{q}},t) = \mathbf{C}^h_q\dot{\mathbf{q}} + \mathbf{C}^h_t = \mathbf{0} \tag{32}$$

Where $\mathbf{C}^h_t$ is the partial derivative of the vector of holonomic constraints with respect to time. This equation, i.e., Eq.(32) can be combined with the nonholonomic constraints of Eq.(30), which represents a direct restriction on the vector of generalized velocities, to yield the following velocity relationships

$$\begin{bmatrix} \mathbf{C}^h_q\dot{\mathbf{q}} + \mathbf{C}^h_t \\ \mathbf{G}(\mathbf{q},t)\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q},t) \end{bmatrix} = \mathbf{0} \tag{33}$$

$$\Updownarrow$$

$$\begin{bmatrix} \mathbf{C}^h_q \\ \mathbf{G}(\mathbf{q},t) \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} \mathbf{C}^h_t \\ \mathbf{g}(\mathbf{q},t) \end{bmatrix} = \mathbf{0} \tag{34}$$

In order to obtain the kinematic relationships between the accelerations, one may differentiate the velocity equations of Eq.(33), as follows:

$$\mathbf{C}_{(\mathbf{q},\dot{\mathbf{q}})}\ddot{\mathbf{q}} = \begin{bmatrix} \mathbf{C}^h_{\mathbf{q}} \\ \mathbf{G} \end{bmatrix} \ddot{\mathbf{q}} = -\begin{bmatrix} \left(\mathbf{C}^h_{\mathbf{q}}\dot{\mathbf{q}}\right)_{\mathbf{q}}\dot{\mathbf{q}} + 2\mathbf{C}^h_{\mathbf{q}t}\dot{\mathbf{q}} + \mathbf{C}_{tt} \\ \left(\mathbf{G}\dot{\mathbf{q}}\right)_{\mathbf{q}}\dot{\mathbf{q}} + \left(\mathbf{g}_{\mathbf{q}} + \mathbf{G}_t\right)\dot{\mathbf{q}} + \mathbf{g}_t \end{bmatrix} = \begin{bmatrix} \mathbf{Q}^h_d \\ \mathbf{Q}^{nh}_d \end{bmatrix} \tag{35}$$

3.1 Velocity Stabilization of systems with Non-Holonomic constraints

1. At each integration step time t, the constraints function derivative, $\dot{\mathbf{C}}(\mathbf{q},t)$ as well as the constraints Jacobian, $\mathbf{C}_{\mathbf{q}}$ should be evaluated. Note that the vector $\dot{\mathbf{C}}(\mathbf{q},t)$ can be obtained using Eq.(2) and $\frac{\partial \dot{\mathbf{C}}(\mathbf{q},t)}{\partial \dot{\mathbf{q}}} = \frac{\partial}{\partial \dot{\mathbf{q}}}\left(\mathbf{C}_q\dot{\mathbf{q}} + \mathbf{C}_t\right) = \mathbf{C}_{\mathbf{q}}$.
2. Evaluate the Newton difference, $\Delta\dot{\mathbf{q}}_n$ as:

$$\begin{aligned} \Delta\dot{\mathbf{q}}_n &= -\begin{bmatrix} \mathbf{C}^h_{\mathbf{q}} \\ \mathbf{C}^{nh}_{\dot{\mathbf{q}}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{C}^h_q\dot{\mathbf{q}} + \mathbf{C}^h_t \\ \mathbf{G}\dot{\mathbf{q}} + \mathbf{g} \end{bmatrix} \\ &= -\begin{bmatrix} \mathbf{C}^h_{\mathbf{q}} \\ \mathbf{C}^{nh}_{\dot{\mathbf{q}}} \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathbf{C}^h_q \\ \mathbf{G} \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} \mathbf{C}^h_t \\ \mathbf{g} \end{bmatrix} \right) \end{aligned}$$

3. Update the generalized velocities as follows:

$$\dot{\mathbf{q}}_{n+1} = \dot{\mathbf{q}}_n + \Delta\dot{\mathbf{q}}_n \tag{36}$$

3.2 Selective Generalized Coordinates Partitioning

In the generalized coordinates partitioning method, described in Sec.(2.3), Gaussian elimination is used to identify the dependent coordinates set, and consequently the remaining set of independent coordinates. The partitioning step is carried out at each time step of the numerical integration. Moreover, it may be implemented more than once to avoid some configuration singularities. It is noticed that the independent coordinates set depends on the joint's type (revolute, prismatic, etc.) and on the number of constraints equations assigned for these joints. For example, if prismatic joint constraints a body to move except translating along the $x-$axis; the independent coordinate mostly will be R_x. However, this observation will be difficult enough for large-scale systems with different nature of bodies and various joints.

In the multibody systems with non-holonomic constraints, if the non-holonomic constraints equations, i.e. Eq. (30), is expressed in terms of some set of generalized velocities, such as

$$\mathbf{C}^{nh}(\mathbf{q},\dot{\mathbf{q}},t) = \mathbf{C}^{nh}(\dot{\theta}^i,t) = \mathbf{0} \tag{37}$$

This means that a set of length n_F of the related coordinates are necessarily independent throughout the rotation of the body i, where n_F is the number of degrees of freedom of the system. According to the dependence on Euler parameters to present the rotational coordinates, the selective integration coordinates can be selected arbitrarily from the Euler parameters' set. In the case of the implementing of this method, the identification of the independent coordinates should be eliminated and consequently; the computational time should be decreased. This method will be examined and evaluated in the following section.

4 SIMULATION OF EPICYCLIC GEAR TRAIN

In this section, an Epicyclic gear train is presented as multibody system subjected to holonomic and nonholonomic con-

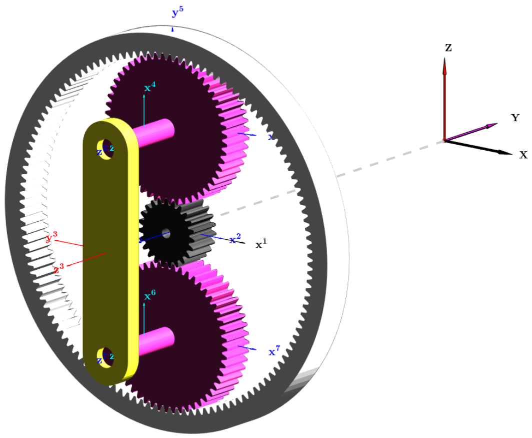

FIGURE 2. Multibody model Epicyclic gear train

straints. The simulation's results are shown, and comparisons between the integration methods are carried out in order to evaluate the effectiveness of the proposed method.

4.1 Description of the Non-holonomic System

The multibody system of the Epicyclic gearbox can be assembled using seven bodies in its simple case. It should be pointed out that, according to our choice of using Euler parameters to designate the body orientation, there will be seven the number of coordinates for each body. The ring gear is hold fixed by rigid joint with the ground. The gearbox, see Fig.(2), have two pins that are rigidly attached to the arm body from one end and carry two planet gears from the other end. The multibody system can be constructed with the arrangement listed in Tab.(1). The total number of generalized coordinates are 49; the dependency between them can be determined by defining the constraints function according to the type of joints. The holonomic constraints can be considered as based on the arrangement of Tab.(2).
In this table, the number of constraints includes the Euler parameters constraints equation for each body, i.e., $\theta^i\theta^{iT} = 1$. Therefore, the total number of holonomic constraints is 45; and thus the system has 4 degrees of freedom. The Non-holonomic constraints result from the fact that the relative velocities of the engaged gears are zero at their contact points (the pitch points). Therefore, the following equations apply:

$$r_3\bar{\omega}_z^3 = r_2\bar{\omega}_z^2 + r_5\bar{\omega}_z^5 \quad (38)$$

$$r_1\bar{\omega}_z^1 = r_3\bar{\omega}_z^3 + r_5\bar{\omega}_z^5 \quad (39)$$

Where $r_3 = (r_2 + r_5)$ is the arm radius and $r_1 = (r_2 + 2r_5)$ is the radius of the ring gear. By using the definition of gear module, i.e., $m = DT$, Eqs.(38, 39) can be written as:

$$T_2\left(\bar{\omega}_z^2 - \bar{\omega}_z^3\right) + T_5\left(\bar{\omega}_z^5 - \bar{\omega}_z^3\right) = 0 \quad (40)$$

$$(T_2 + 2T_5)\,\bar{\omega}_z^1 - (T_2 + T_5)\,\bar{\omega}_z^3 - T_5\bar{\omega}_z^5 = 0 \quad (41)$$

Where T_i is the number of teeth of respective gear body and $\bar{\omega}_z^i$ is the angular velocity about its local $z-$ axis.

TABLE 1. Multibody system components

#.	Body Name	Index of coords	Inertia Properties $[Kg], [Kg.m^2]$			
			m	I_{xx}	I_{yy}	I_{zz}
1	*Ring*	$1 \to 7$	6.97	0.25	0.255	0.507
2	*Sun*	$8 \to 14$	0.85	0.00064	0.00064	0.00087
3	*Arm*	$15 \to 21$	5.42	0.0032	0.0569	0.05957
4	*Carr*1	$22 \to 28$	0.832	0.0016	0.0016	0.00093
5	*Plnt*1	$29 \to 35$	4.41	0.013	0.013	0.024
6	*Carr*2	$36 \to 42$	0.832	0.0016	0.0016	0.00093
7	*Plnt*2	$43 \to 49$	4.41	0.013	0.013	0.024

TABLE 2. Holonomic Constraints

Joint Type	$Body(i)$	$Body(j)$	*# constraints eqs.*
Rigid	*Ground*	*Ring*	7
Revolute	*Ground*	*Sun*	6
Revolute	*Ground*	*Arm*	6
Rigid	*Arm*	*Carrier*1	7
Revolute	*Carrier*1	*Planet*1	6
Rigid	*Arm*	*Carrier*2	7
Revolute	*Carrier*2	*Planet*2	6

One more constraint equation can be added to the second planet gear, to relate the rotational speed of the arm, planet and sun gears, such that

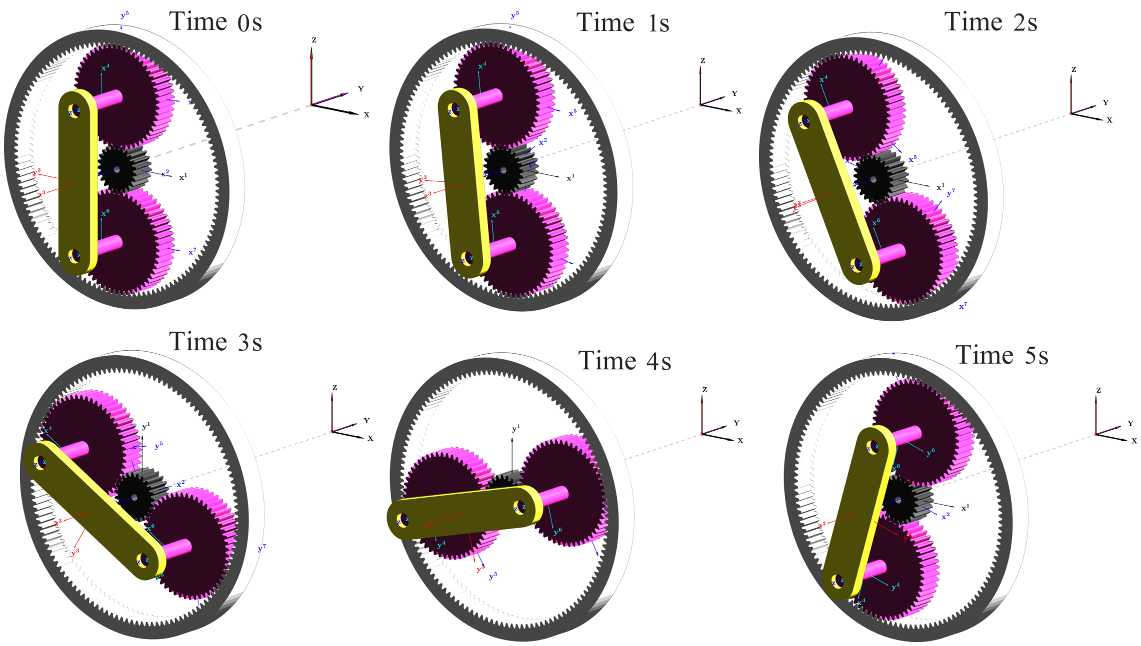

FIGURE 3. Time Frames of system animation through 5[s] simulation time

$$T_2\left(\bar{\omega}_z^2-\bar{\omega}_z^3\right)+T_7\left(\bar{\omega}_z^7-\bar{\omega}_z^3\right)=0 \tag{42}$$

Thus, the total number holonomic and non-holonomic constraints becomes 48 and consequently, the system has only one degree of freedom. The simulation results in the following section are carried out by applying an exerted torque of 0.1 $[N.m]$ on the arm body along with the rotational axis (local z^3-axis). The system parameters are as follows: $m=4[mm]$, $T_1=127, T_2=23, T_{5,7}=52$ and the gear ratio is 6.41.

4.2 Comparisons Between Integration Methods

In order to evaluate the proposed scheme for integrating selected coordinates along with the simulation time; so we need to compare the simulation results of the selective coordinates partitioning with full-coordinates integration, see Sec.(2), and with the results of the generalized coordinates partitioning, see Sec.(2.3). These methods can be referred as FCI, GCPI. and SCPI. The integration process carried out using the MATLAB code, *ode113*, which is a variable step variable order method uses Adams–Bashforth–Moulton predictor-correctors of order 1 to 13 [13]. The relative and absolute error tolerances are selected to be 0.001 for all methods. The main functions which are computed throughout the integration process are as follows:

1. ODE function (f1), that constructs the state vector to be integrated. In this function, the acceleration vector is computed, see Eq.(4) for FCI method. In the case of GCPI and SCPI methods, the acceleration vector will be the output vector from the next function.
2. Coordinate partitioning function (f2), in which the MATLAB code, *rref* (Reduced row echelon form), is used to identify the independent coordinates set. Furthermore, this function constructs the acceleration vector based on Eq.(23). This function is used only for CPI schemes. In the SCPI method, the use of *rref* function is eliminated, and only one rotational parameter of the body number (2/3/5 or 7) can be selected as an independent coordinate for integration. This selection is based on Eq. (37) and Eqs.(40 - 42) representing the non-holonomic constraints of the system.
3. Position stabilization function (f3) that adjust the numerical results in the position level as discussed in Sec.(2.1).
4. Holonomic velocity stabilization function (f4) that adjust the numerical results in the velocity level of systems with holonomic constraints as discussed in Sec.(2.2).
5. Non-Holonomic velocity stabilization function (f5) that readjust the numerical results in the velocity level as discussed in Sec.(3.1).

Although there are many integration codes with higher accuracy available in the MATLAB software, as well as more adequate coordinate reduction strategies [7, 14, 15]; the main objective here is to determine the effectiveness of the SCPI method.

The numerical integration of the system described in

Sec.(4.1) is carried out by the three methods for a simulated time of 15[s]. The results of FCI method fall within the acceptable limits of the error tolerances and constraints' preciseness up to simulation time of approximately 12[s]. Figure (3) shows the animation frames of the system during the first 5[s] of the time simulation. The non-holonomic constraints' preciseness, which can be defined as $\eta^{nh} = \left\|\mathbf{C}^{nh}(\mathbf{q}, \dot{\mathbf{q}})\right\|$, is shown in Fig.(4); also, the angular velocities of gears are plotted in Fig.(5). Regardless the sharp change in the system angular velocities; the ascending and accumulative behavior of the errors is not acceptable. This method suffers from an accumulation of constraints error and may produce a strong violation of the position and velocity constraint equations. For moderate size multibody system dynamics applications and small intervals of simulation time, this method may be satisfactory.

The results of both GCPI and SCPI methods are coincided with each other and fulfilled the limits of the error tolerances and constraints' preciseness with acceptable margins for the complete simulation period. The non-holonomic constraints' preciseness, η^{nh}, is plotted in Fig.(6) with clear stabilizing errors and acceptable values. The corresponding angular velocities of gears are shown in Fig.(7). Constant torque exerted on the rigid body, result in a linear increasing speed, consequently, the curves of Fig.(7) do not necessitate an evaluation.

The implementation of the GCPI has been found to be reliable and accurate. However, it may suffer from worse numerical efficiency due to the requirement for the iterative solution of the dependent generalized coordinates as well as the Gaussian elimination algorithm to identify the independent set of generalized coordinates.

Regarding the implications of integration time calculation, it has been monitoring the following table based on numerical simulations process for 15[s] of simulation time interval:

#	*Calculation time*
FCI	216[s]
GCPI	229[s]
SCPI	54.37[s]

The table shows significant decrease of the calculation time of the SCPI compared with other methods. This calculation time is the total time consumed in all functions calculation, including the five functions (f1) to (f5) and other functions to calculate the forces and quadratic terms of the equations of motion. Figure (8) shows a comparison between the self computational time of the main five functions, the self time is the time spent in a function excluding the time spent in its child functions. The comparison shows a dramatic decrease in the computational time of the Coordinate partitioning function (f2) with more than seven times. It is found that this ratio is not affected by the selected set of the integration coordinates.

Figures (9 - 11) shows the simulation of Euler parameters

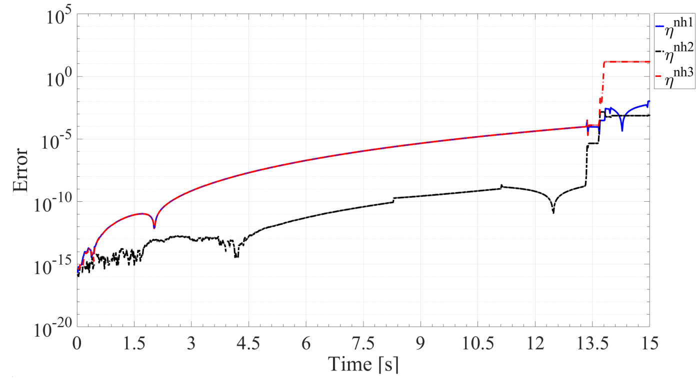

FIGURE 4. Non-holonomic constraints error of FCI method

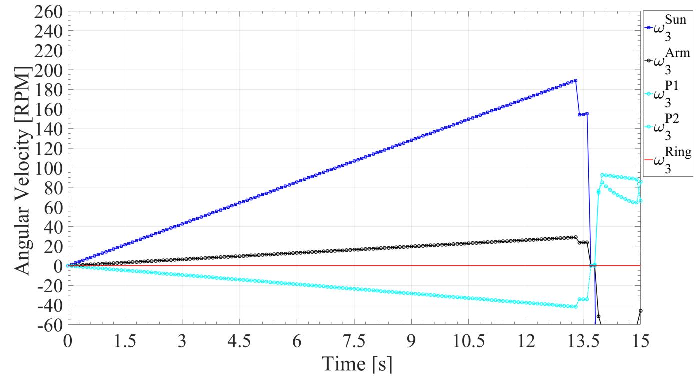

FIGURE 5. Simulation of system angular velocities using FCI method

for the rotating bodies, which are related to each other with some relative generalized velocities. The figures show that the Euler parameters for each body satisfy the constraint of $\theta^i\theta^{iT} = 1$ and all the orientation coordinates varies smoothly within the time range.

SUMMARY AND CONCLUSIONS

In this paper, the generalized coordinates partitioning algorithm is extended to define dependent generalized velocity coordinates associated with nonholonomic constraints. Furthermore, a method for selective generalized coordinates partitioning is proposed for integrating multibody system subjected to nonholonomic constraints. Based on the simulation work of planetary gear system, and the comparison between full-coordinates integration and the coordinates partitioning methods, it can be concluded that the proposed method of selecting a set of independent coordinates for numerical integration is a powerful and accurate enough for non-holonomic systems. The set of selective generalized coordinates is assigned according to the nonholonomic constraints functions and its related generalized ve-

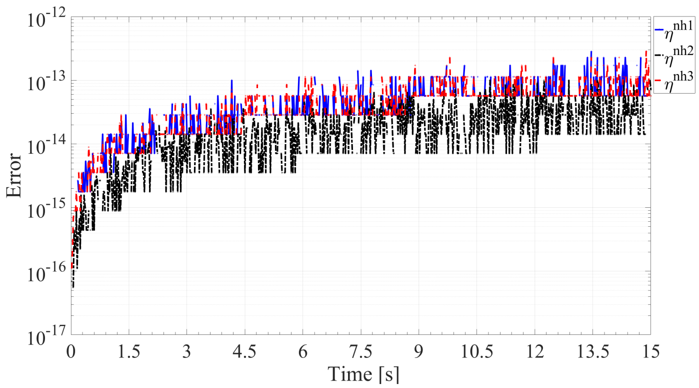

FIGURE 6. Non-holonomic Constraints error of GCPI and SCPI methods

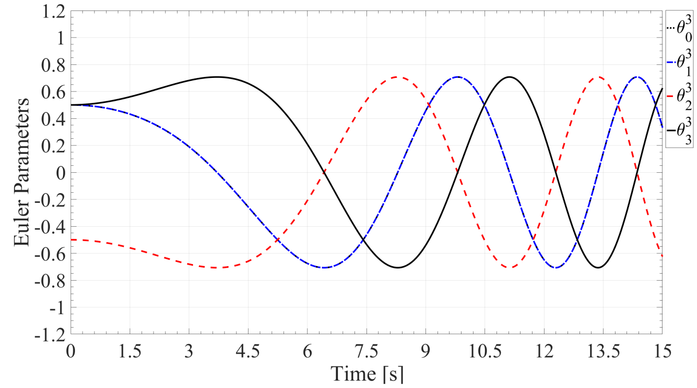

FIGURE 9. Simulation of Euler parameters of the arm body

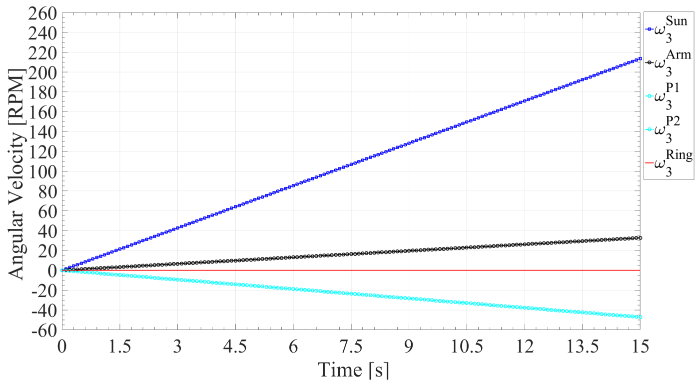

FIGURE 7. Simulation of system angular velocities using CPI methods

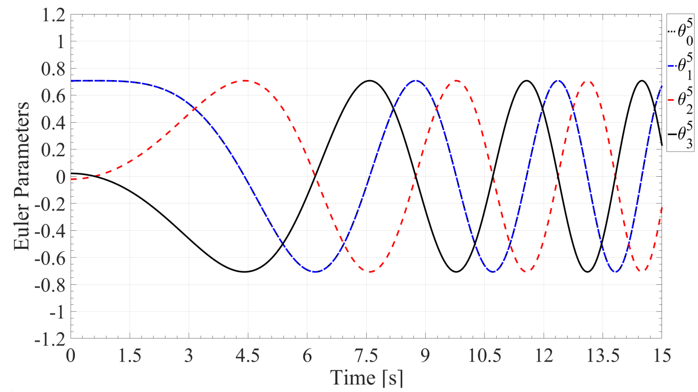

FIGURE 10. Simulation of Euler parameters of the planet gears

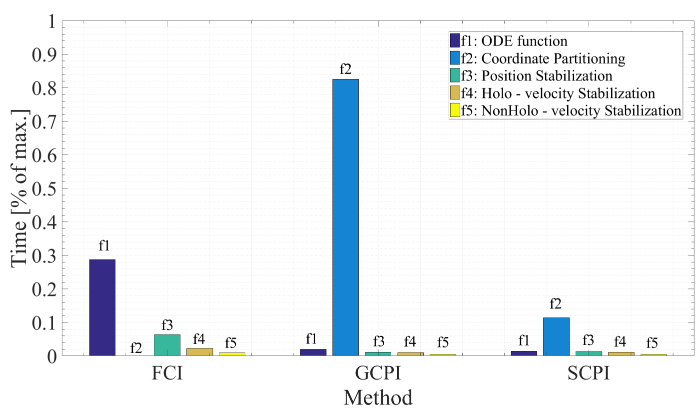

FIGURE 8. Calculation time per function for integration methods

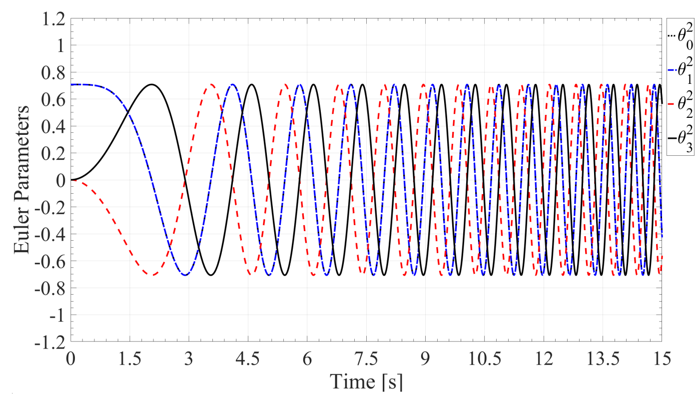

FIGURE 11. Simulation of Euler parameters of the sun gear

locities.This method adds significant features to the generalized coordinates partitioning regarding to the calculation time of the simulation process. It contributes a reduction of calculation time more than seven times. With this in mind, it retains the control over error rates during the integration process.

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REFERENCES

[1] Sany, J.R., Shabana, A.A., 1989, ”On the numerical solution of differential/algebraic equations of motion of deformable mechanical systems with nonholonomic constraints”, Computers & Structures,**33**(4), pp.1017-1029.

[2] Wehage, R.A. and Haug, E.J., 1982, ”Generalized Coordinate Partitioning for Dimension Reduction in Analysis of Constrained Dynamic Systems”, ASME Journal of Mechanical Design, **134**: pp. 247-255.

[3] Shabana, A., 1994, *Computational Dynamics*, John Wiley & Sons.

[4] Shabana, Ahmed A., 2005, *Dynamics of Multibody Systems*. 3rd ed., Cambridge University Press UK.

[5] Nada, A.A., Hussein, B.A., Megahed, S.M., and Shabana, A.A., 2009, ”Floating Frame of Reference and Absolute Nodal Coordinate Formulations in the Large Deformation Analysis of Robotic Manipulators: A Comparative Experimental and Numerical Study”, ASME(IDETC/CIE 2009), San Diego, California, USA.

[6] Aboubakr, A.K., Shabana, A.A., 2015, ”Efficient and robust implementation of the TLISMNI method”, Journal of Sound and Vibration, **353**, pp. 220-242.

[7] Haug, E.J., 2016, ”An index 0 differential-algebraic equation formulation for multibody dynamics: Nonholonomic constraints”, Journal of Mechanics Based Design of Structures and Machines, doi: 10.1080/15397734.2016.1273783

[8] Shabana, A.A., 2014, ”Euler parameters kinetic singularity”, Proceedings of the Institution of Mechanical Engineers, Part K: Journal of Multi-body, **228**(3), pp.307-313.

[9] Korkealaakso, P., Mikkola, A., Rantalainen, T., Rouvinen, A., 2009, ”Description of joint constraints in the floating frame of reference formulation”, Proceedings of the Institution of Mechanical Engineers, Part K: Journal of Multibody Dynamics, **223**(2), pp. 133-145.

[10] Nada, A. A., Hussein, B. A., Megahed,S.M., Shabana A.A., 2010. ”Use of the floating frame of reference formulation in large deformation analysis: experimental and numerical validation”. Proceedings of the Institution of Mechanical Engineers, Part K: Journal of Multi-body Dynamic **224**, pp.45-58.

[11] Nada, A. A., 2007, ”Flexible Robotic Manipulators: Modeling, Simulation and Control with Experimentation”, PhD thesis. In: Mechanical Design and Production Engineering, Cairo University, Egypt.

[12] de Falco, D., Pennestri, E., Vita, L., 2009, ”Investigation of the Influence of Pseudoinverse Matrix Calculations on Multibody Dynamics Simulations by Means of the Udwadia-Kalaba Formulation”, Journal of aerospace engineering, ASCE, **22**(4), pp. 365-372.

[13] Shampine, L.F. and Reichelt, M.W., 1997, ”The Matlab ODE suite”, SIAM J. Sci. Comput., **18**(1), pp.1-22.

[14] Kim, S.S., and Vanderploeg, M.J., 1985, ”QR Decomposition for State Space Representation of Constrained Mechanical Dynamic Systems”, Journal of Journal of Mechanisms, Transmissions, and Automation in Design, **108**(2), pp.183-188.

[15] Singh, R.P. and Likins, P.W., 1985, ”Singular Value Decomposition for Constrained Dynamical Systems”, Journal of Applied Mechanics, **52**(4), pp. 943-948.